

Time-Dependent Lagrangian Biomechanics

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Abstract

In this paper we present the time-dependent generalization of an ‘ordinary’ autonomous human musculo-skeletal biomechanics. We start with the configuration manifold of human body, given as a set of its all active degrees of freedom (DOF). This is a Riemannian manifold with a material metric tensor given by the total mass-inertia matrix of the human body segments. This is the base manifold for standard autonomous biomechanics. To make its time-dependent generalization, we need to extend it with a real time axis. On this extended configuration space we develop time-dependent biomechanical Lagrangian dynamics, using derived jet spaces of velocities and accelerations, as well as the underlying geometric evolution of the mass-inertia matrix.

Keywords: Human time-dependent biomechanics, configuration manifold, jet spaces, geometric evolution

1 Introduction

It is a well-known fact that most of dynamics in both classical and quantum physics is based on *assumption of a total energy conservation* (see, e.g. [1]). Dynamics based on this assumption of time-independent energy, usually given by Lagrangian or Hamiltonian energy function, is called *autonomous*. This basic assumption is naturally inherited in human biomechanics, formally developed using Newton–Euler, Lagrangian or Hamiltonian formalisms (see [2, 3, 4, 5, 6, 7, 8, 9]). And this works fine for most individual movement simulations and predictions, in which the total human energy dissipations are insignificant. However, if we analyze a 100 m-dash sprinting motion, which is in case of top athletes finished under 10 s, we can recognize a significant slow-down after about 70 m in *all* athletes – despite of their strong intention to finish and win the race, which is an obvious sign of the total energy dissipation. In other words, the *total mechanical + biochemical energy* of a sprinter cannot be conserved even for 10 s. So, if we want to develop a realistic model of intensive human motion that is longer than 7–8 s (not to speak for instance of a 4 hour tennis match), we necessarily need to use the more advanced formalism of time-dependent mechanics.

Similarly, if we analyze individual movements of gymnasts, we can clearly see that the high speed of these movements is based on quickly-varying mass-inertia distribution of various body segments (mostly arms and legs). Similar is the case of pirouettes in ice skating. As the total mass-inertia matrix M_{ij} of a biomechanical system corresponds to the Riemannian metric tensor g_{ij} of its configuration manifold, we can formulate this problem in terms of time-dependent Riemannian geometry [4, 1].

The purpose of this paper is to introduce the time-dependent biomechanics (with its underlying geometric evolution) to the general applied-mechanics readership. Briefly, we are modeling a complex mechanical system with a time-varying Lagrangian energy function.

2 Biomechanical Manifold and its (Co)Tangent Bundles

2.1 Configuration Manifold of Autonomous Biomechanics

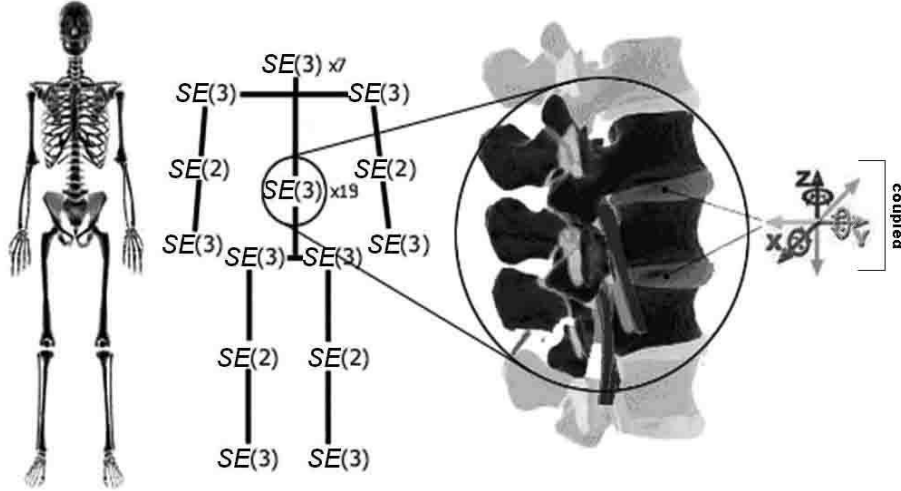


Figure 1: The configuration manifold Q of the human musculo-skeletal dynamics is defined as a topological product of constrained $SE(3)$ groups acting in all major (synovial) human joints, $Q = \prod_k SE(3)^k$ (see [6]).

From precise biomechanical perspective, human body represents a system of flexibly coupled rigid bodies, each with its own 6 DOF, three macro-rotations $x_{\text{rot}}^k(t)$, $k = 1, 2, 3$, plus three micro-translations $x_{\text{trn}}^k(t)$, $k = 1, 2, 3$. More formally, human musculo-skeletal dynamics is defined (see [6]) in terms of Euclidean $SE(3)$ -groups of full rigid-body motion¹ in all main human joints (see Figure 1). Therefore, the *configuration manifold* Q for human musculo-skeletal dynamics is defined as a Cartesian product of all included constrained $SE(3)$ groups, $Q = \prod_j SE(3)^j$ where j labels the active joints. The configuration manifold

¹Briefly, the Euclidean $SE(3)$ -group is defined as a semidirect (noncommutative) product (denoted by \triangleright) of 3D rotations and 3D translations (for technical details see [6, 4, 10, 1]).

Q is coordinated by local joint coordinates $x^i(t)$, $i = 1, \dots, n = \text{total number of active DOF}$. The corresponding joint velocities $\dot{x}^i(t)$ live in the *velocity phase space* TQ , which is the tangent bundle of the configuration manifold Q .

The velocity phase space TQ has the Riemannian geometry with the *local metric form*:

$$\langle g \rangle \equiv ds^2 = g_{ij} dx^i dx^j, \quad (\text{Einstein's summation convention is in use})$$

where $g_{ij}(x)$ is the material metric tensor defined by the biomechanical system's *mass-inertia matrix* and dx^i are differentials of the local joint coordinates x^i on Q . Besides giving the local distances between the points on the manifold Q , the Riemannian metric form $\langle g \rangle$ defines the system's kinetic energy:

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

giving the *Lagrangian equations* of the conservative skeleton motion with kinetic-minus-potential energy Lagrangian $L = T - V$, with the corresponding *geodesic form* [8]

$$\frac{d}{dt} L_{\dot{x}^i} - L_{x^i} = 0 \quad \text{or} \quad \ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0,$$

where subscripts denote partial derivatives, while Γ_{jk}^i are the Christoffel symbols of the affine Levi-Civita connection of the biomechanical manifold Q .

This is the basic geometrical structure for *autonomous Lagrangian biomechanics*. In the next section will extend this basic structure to embrace the time-dependent biomechanics.

3 Biomechanical Jets

While standard autonomous Lagrangian biomechanics is developed on the configuration manifold Q , the *time-dependent biomechanics* necessarily includes also the real time axis \mathbb{R} , so we have an *extended configuration manifold* $\mathbb{R} \times Q$. Slightly more generally, the fundamental geometrical structure is the so-called *configuration bundle* $\pi : Q \rightarrow \mathbb{R}$. Time-dependent biomechanics is thus formally developed either on the *extended configuration manifold* $\mathbb{R} \times Q$, or on the configuration bundle $\pi : Q \rightarrow \mathbb{R}$, using the concept of *jets*, which are based on the idea of *higher-order tangency*, or higher-order contact, at some designated point (i.e., certain joint) on a biomechanical configuration manifold Q .

Formally, a pair of smooth manifold maps, $f_1, f_2 : M \rightarrow N$ (see Figure 2), are said to be *k-tangent* (or *tangent of order k*, or have a *kth order contact*) at a point x on a domain manifold M , denoted by $f_1 \sim f_2$, iff

$$\begin{aligned} f_1(x) &= f_2(x) && \text{called } 0\text{-tangent,} \\ \partial_x f_1(x) &= \partial_x f_2(x), && \text{called } 1\text{-tangent,} \\ \partial_{xx} f_1(x) &= \partial_{xx} f_2(x), && \text{called } 2\text{-tangent,} \\ &\dots && \text{etc. to the order } k \end{aligned}$$

In this way defined *k-tangency* is an *equivalence relation*.

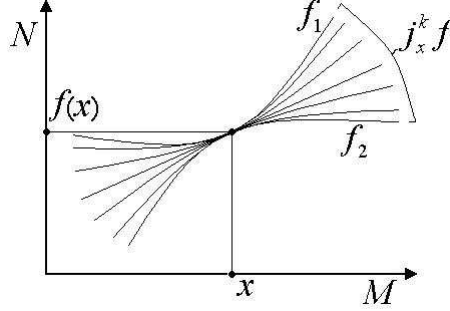


Figure 2: An intuitive geometrical picture behind the k -jet concept, based on the idea of a higher-order tangency (or, higher-order contact).

A k -jet (or, a *jet of order k*), denoted by $j_x^k f$, of a smooth map $f : Q \rightarrow N$ at a point $x \in Q$ (see Figure 2), is defined as an *equivalence class* of k -tangent maps at x ,

$$j_x^k f : Q \rightarrow N = \{f' : f' \text{ is } k\text{-tangent to } f \text{ at } x\}.$$

For example, consider a simple function $f : X \rightarrow Y$, $x \mapsto y = f(x)$, mapping the X -axis into the Y -axis in \mathbb{R}^2 . At a chosen point $x \in X$ we have:

a 0-jet is a graph: $(x, f(x))$;

a 1-jet is a triple: $(x, f(x), f'(x))$;

a 2-jet is a quadruple: $(x, f(x), f'(x), f''(x))$,

and so on, up to the order k (where $f'(x) = \frac{df(x)}{dx}$, etc).

The set of all k -jets from $j_x^k f : X \rightarrow Y$ is called the k -jet manifold $J^k(X, Y)$.

Now, given the biomechanical configuration bundle $Q \rightarrow \mathbb{R}$ over the time axis \mathbb{R} , the 1-jet space $J^1(\mathbb{R}, Q)$ is the set of equivalence classes $j_t^1 s$ of sections $s^i : \mathbb{R} \rightarrow Q$ of the configuration bundle $Q \rightarrow \mathbb{R}$, which are identified by their values $s^i(t)$, as well as by the values of their partial derivatives $\partial_t s^i = \partial_t s^i(t)$ at time points $t \in \mathbb{R}$. The 1-jet manifold $J^1(\mathbb{R}, Q)$ is coordinated by (t, x^i, \dot{x}^i) , that is by (*time, coordinates and velocities*) at every active human joint, so the 1-jets are local joint coordinate maps

$$j_t^1 s : \mathbb{R} \rightarrow Q, \quad t \mapsto (t, x^i, \dot{x}^i).$$

Similarly, the 2-jet space $J^2(\mathbb{R}, Q)$ is the set of equivalence classes $j_t^2 s$ of sections $s^i : \mathbb{R} \rightarrow Q$ of the configuration bundle $\pi : Q \rightarrow \mathbb{R}$, which are identified by their values $s^i(t)$, as well as the values of their first and second partial derivatives, $\partial_t s^i = \partial_t s^i(t)$ and $\partial_{tt} s^i = \partial_{tt} s^i(t)$, respectively, at time points $t \in \mathbb{R}$. The 2-jet manifold $J^2(\mathbb{R}, Q)$ is coordinated by $(t, x^i, \dot{x}^i, \ddot{x}^i)$, that is by (*time, coordinates, velocities and accelerations*) at every active human joint, so the 2-jets are local joint coordinate maps²

$$j_t^2 s : \mathbb{R} \rightarrow Q, \quad t \mapsto (t, x^i, \dot{x}^i, \ddot{x}^i).$$

²For more technical details on jet spaces with their physical applications, see [16, 17, 18, 19]).

4 Lagrangian Time-Dependent Biomechanics

The general form of time-dependent Lagrangian biomechanics with *time-dependent Lagrangian* function $L(t; x^i; \dot{x}^i)$ defined on the jet space $X = J^1(\mathbb{R}, Q) \cong \mathbb{R} \times TQ$, with local canonical coordinates: $(t; x^i; \dot{x}^i) = (\text{time, coordinates and velocities})$ in active local joints, can be formulated as [4, 1]

$$\frac{d}{dt}L_{\dot{x}^i} - L_{x^i} = \mathcal{F}_i(t, x, \dot{x}), \quad (i = 1, \dots, n), \quad (1)$$

where the coordinate and velocity partial derivatives of the Lagrangian are respectively denoted by L_{x^i} and $L_{\dot{x}^i}$.

4.1 Local Muscle–Joint Mechanics

The right-hand side terms $\mathcal{F}_i(t, x, \dot{x})$ of (1) denote any type of external torques and forces, including excitation and contraction dynamics of muscular–actuators and rotational dynamics of hybrid robot actuators, as well as (nonlinear) dissipative joint torques and forces and external stochastic perturbation torques and forces. In particular, we have [2, 3]:

1. Synovial joint mechanics, giving the first stabilizing effect to the conservative skeleton dynamics, is described by the (x, \dot{x}) -form of the Rayleigh – Van der Pol’s dissipation function

$$R = \frac{1}{2} \sum_{i=1}^n (\dot{x}^i)^2 [\alpha_i + \beta_i (x^i)^2],$$

where α_i and β_i denote dissipation parameters. Its partial derivatives give rise to the viscous-damping torques and forces in the joints

$$\mathcal{F}_i^{joint} = \partial R / \partial \dot{x}^i,$$

which are linear in \dot{x}^i and quadratic in x^i .

2. Muscular mechanics, giving the driving torques and forces $\mathcal{F}_i^{muscle} = \mathcal{F}_i^{muscle}(t, x, \dot{x})$ with $(i = 1, \dots, n)$ for human biomechanics, describes the internal excitation and contraction dynamics of equivalent muscular actuators [11].

(a) *Excitation dynamics* can be described by an impulse force–time relation

$$\begin{aligned} F_i^{imp} &= F_i^0 (1 - e^{-t/\tau_i}) & \text{if stimulation} > 0 \\ F_i^{imp} &= F_i^0 e^{-t/\tau_i} & \text{if stimulation} = 0, \end{aligned}$$

where F_i^0 denote the maximal isometric muscular torques and forces, while τ_i denote the associated time characteristics of particular muscular actuators. This relation represents a solution of the Wilkie’s muscular active–state element equation [12]

$$\dot{\mu} + \gamma \mu = \gamma S A, \quad \mu(0) = 0, \quad 0 < S < 1,$$

where $\mu = \mu(t)$ represents the active state of the muscle, γ denotes the element gain, A corresponds to the maximum tension the element can develop, and $S = S(r)$ is the ‘desired’ active state as a function of the motor unit stimulus rate r . This is the basis for biomechanical force controller.

(b) *Contraction dynamics* has classically been described by the Hill's hyperbolic force-velocity relation [13]

$$F_i^{Hill} = \frac{(F_i^0 b_i - \delta_{ij} a_i \dot{x}^j)}{(\delta_{ij} \dot{x}^j + b_i)},$$

where a_i and b_i denote the Hill's parameters, corresponding to the energy dissipated during the contraction and the phosphagenic energy conversion rate, respectively, while δ_{ij} is the Kronecker's δ -tensor.

In this way, human biomechanics describes the excitation/contraction dynamics for the i th equivalent muscle-joint actuator, using the simple impulse-hyperbolic product relation

$$\mathcal{F}_i^{muscle}(t, x, \dot{x}) = F_i^{imp} \times F_i^{Hill}.$$

Now, for the purpose of biomedical engineering and rehabilitation, human biomechanics has developed the so-called *hybrid rotational actuator*. It includes, along with muscular and viscous forces, the D.C. motor drives, as used in robotics [14, 2]

$$\begin{aligned} \mathcal{F}_k^{robo} &= i_k(t) - J_k \ddot{x}_k(t) - B_k \dot{x}_k(t), \quad \text{with} \\ l_k i_k(t) + R_k i_k(t) + C_k \dot{x}_k(t) &= u_k(t), \end{aligned}$$

where $k = 1, \dots, n$, $i_k(t)$ and $u_k(t)$ denote currents and voltages in the rotors of the drives, R_k, l_k and C_k are resistances, inductances and capacitances in the rotors, respectively, while J_k and B_k correspond to inertia moments and viscous dampings of the drives, respectively.

Finally, to make the model more realistic, we need to add some stochastic torques and forces [15]

$$\mathcal{F}_i^{stoch} = B_{ij}[x^i(t), t] dW^j(t),$$

where $B_{ij}[x(t), t]$ represents continuous stochastic diffusion fluctuations, and $W^j(t)$ is an N -variable Wiener process (i.e., generalized Brownian motion), with

$$dW^j(t) = W^j(t + dt) - W^j(t), \quad (\text{for } j = 1, \dots, N).$$

4.2 Time-Dependent Riemannian Geometry

As illustrated in the introduction, the mass-inertia matrix of human body, defining the Riemannian metric tensor $g_{ij}(x)$ need not be time-constant, as in case of fast gymnastic movements and pirouettes in ice skating, which are based on quick variations of inertia moments and products constituting the material metric tensor $g_{ij}(x)$. In general, the metric tensor g_{ij} is both time and joint dependent, $g_{ij} = g_{ij}(t, x)$. This time-dependent Riemannian geometry can be formalized in terms of the *Ricci flow* [20], the nonlinear heat-like evolution metric equation:

$$\partial_t g_{ij} = -R_{ij}, \quad (2)$$

for a time-dependent Riemannian metric $g = g_{ij}(t)$ on a smooth n -manifold Q with the Ricci curvature tensor R_{ij} . This equation roughly says that we can deform any metric on the configuration manifold Q by the negative of its curvature; after *normalization*, the final state of such deformation will be a metric with constant curvature. The negative sign in

(2) insures a kind of global *volume exponential decay*,³ since the Ricci flow equation (2) is a kind of nonlinear geometric generalization of the standard linear *heat equation*

$$\partial_t u = \Delta u.$$

This exponentially-decaying geometrical diffusion is a formal description for pirouettes in ice skating.

5 Conclusion

In this paper we have presented time-dependent generalization of an ‘ordinary’ autonomous human musculo-skeletal biomechanics. Firstly, we have defined the basic configuration manifold Q of human musculo-skeletal biomechanics as an anthropomorphic chain of constrained Euclidean motion groups $SE(3)$. Secondly, we have extended this base manifold by the real time axis \mathbb{R} and time-dependent defined Lagrangian dynamics using first and second order jet spaces. Then we moved to time-dependent Riemannian geometry and showed that the exponential-like decay of total biomechanical energy (due to exhaustion of biochemical resources) is closely related to the Ricci-flow based geometrical diffusion.

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³This complex geometric process is globally similar to a generic exponential decay ODE:

$$\dot{x} = -\lambda f(x),$$

for a positive function $f(x)$. We can get some insight into its solution from the simple exponential decay ODE,

$$\dot{x} = -\lambda x \quad \text{with the solution} \quad x(t) = x_0 e^{-\lambda t},$$

where $x = x(t)$ is the observed quantity with its initial value x_0 and λ is a positive decay constant.

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